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The *K*th-Best Approach for Linear Bilevel Multi-follower Programming

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Abstract. The majority of research on bilevel programming has centered on the linear version of the problem in which only one leader and one follower are involved. This paper addresses linear bilevel multi-follower programming (BLMFP) problems in which there is no sharing information among followers. It explores the theoretical properties of linear BLMFP, extends the *K*th-best approach for solving linear BLMFP problems and gives a computational test for this approach.

Key words: bilevel decision-making, decision-making optimization, *K*th-best approach, linear bilevel programming, multi-follower programming

1. Introduction

A bilevel programming (BLP) problem can be viewed as the noncooperative, two-player game introduced by Von Stackelberg (Stackelberg, 1952) in the context of unbalanced economic markets. In a basic BLP model, the control for the decision variables is partitioned amongst the players. The upper-level is termed as the leader and the lower-level is termed as the follower. The follower makes its decisions after, and in full view of, the leader's decision. Because the set of feasible choices available to either level is interdependent, the leader's decision affects both the follower's payoff and allowable actions, and vice versa (Bard, 1998).

The majority of research on bilevel programming has centered on the linear version of the problem. There have been nearly two dozen algorithms, such as, the *K*th-best approach (Candler and Townsley, 1982), (Bialas and Karwan, 1984), Kuhn-Tucker approach (Bard Falk, 1982), (Bialas and Karwan, 1978), (Hansen, 1992) and (Bialas, 1980), penalty function approach (Aiyoshi and Shimizu, 1981), and (White and Anandalingam, 1993), proposed for solving linear BLP problems since the field caught the attention of researchers in the mid-1970s.

Every linear BLP problem with a finite optimal solution shares the important property that at least one optimal (global) solution is attained at an extreme point of the constraint region. This result was first established by Candler and Townsley (1982) for linear BLP problems with no upper level constraints and with unique lower-level solutions. Afterwards Bard (1984) and Bialas and Karwan (1984) proved this result under the assumption that the constraint region is bounded. The result for the case where upper-level constraints exist has been established by Savard (1989) under no particular assumptions. Based on this property, Candler and Townsley (1982) and Bialas and Karwan (1984) have proposed, respectively, the Kth-best approach that compute global solutions of linear BLP problems by enumerating the extreme points of the constraint region. The Kth-best approach has been proven to be a valuable analysis tool with a wide range of successful applications for linear BLP (Bard, 1998).

Our previous work presented a new definition of solution and related theorem for linear BLP, thus overcame the fundamental deficiency of existing linear BLP theory (Shi et al., 2005a). We also described theoretical properties of linear BLP, developed an extended Kth-best approach for linear BLP (Shi et al., 2005b) an extended Kuhn-Tucker approach and its algorithm for linear BLP (Shi et al., 2005). In (Lu et al. 2005), we proposed a model for linear bilevel multi-follower programming (BLMFP) and a Kuhn-Tucker approach for BLMFP. The intent of this paper is to explore theoretical properties of liner BLMFP, and develop the Kth-best approach for linear BLMFP. Following the introduction, this paper reviews linear bilevel multi-follower programming in Section 2. Theoretical properties and the Kth-best approach are addressed in Section 3. A numeric example for the Kth-best approach is given in Section 4. Section 5 gives a computational test for this approach. Section 6 discusses conclusions and further study.

2. Linear Bilevel Multi-follower Programming

For $x \in X \subset R^n$, $y_i \in Y_i \subset R^{m_i}$, $F : X \times Y_1 \times \cdots \times Y_K \to R^1$, and $f_i : X \times Y_i \to R^1, i = 1, 2, \ldots, K$, a linear BLMFP problem in which $K (K \geqslant 2)$ followers are involved and there is no sharing information among them except the leader's is given (Lu, et al. submitted):

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s \tag{1a}$$

$$\text{subject to } Ax + \sum_{t=1}^{K} B_t y_t \leqslant b \tag{1b}$$

$$\min_{y_i \in Y_i} f_i(x, y_i) = c_i x + e_i y_i \tag{1c}$$

$$\text{subject to } A_i x + C_i y_i \leqslant b_i \tag{1d}$$

where $c \in R^n, c_i \in R^n, d_i \in R^{m_i}, e_i \in R^{m_i}, b \in R^p, b_i \in R^{q_i}, A \in R^{p\times n}, \ B_i \in R^{p\times m_i}, A_i \in R^{q_i\times n}, C_i \in R^{q_i\times m_i}, i = 1, 2, \ldots, K$. As there is not sharing variable among followers, all followers have individual objective function and constraint.

DEFINITION 1. A topological space is compact if every open cover of the entire space has a finite subcover. For example, $[a, b]$ is compact in R (the Heine-Borel theorem, University of Cambridge. http://thesaurus.maths.-org/dictionary/map/word/10037, 2001).

Corresponding to (1) (Lu et al., 2005) gave the following basic definition for linear BLMFP solution.

DEFINITION 2.

(a) Constraint region of the linear BLMFP problem:

$$S = \Big\{(x, y_1, \ldots, y_K) \in X \times Y_1 \times \cdots \times Y_k, \, Ax + \sum_{t=1}^{K} B_t y_t \leqslant b, \\ A_i x + C_i y_i \leqslant b_i, i = 1, 2, \ldots, K\Big\}.$$

The linear BLFMP problem constraint region refers to all possible combinations of choices that the leader and followers may make.

(b) Projection of S onto the leader's decision space:

$$S(X) = \Big\{x \in X : \exists y_i \in Y_i, Ax + \sum_{t=1}^{K} B_t y_t \leqslant b, A_i x + C_i y_i \leqslant b_i, \quad i = 1, 2, \ldots, K\Big\}.$$

Unlike the rules in uncooperative game theory where each player must choose a strategy simultaneously, the definition of BLMFP model requires that the leader moves first by selecting an x in attempting to minimize his objective subjecting to constraints of both upper and each lower level.

(c) Feasible set for each follower $\forall x \in S(X)$:

$$S_i(x) = \{y_i \in Y_i : (x, y_1, \ldots, y_K) \in S\}, i = 1, 2, \ldots, K.$$

The feasible region for the follower is affected by the leader's choice of x, and allowable choices of each follower are the elements of S.

(d) Each follower's rational reaction set for$x \in S(X)$:

$$P_i(x) = \Big\{y_i \in Y_i : y_i \in \arg\min[f_i(x, \hat{y}_i) : \hat{y}_i \in S_i(x)]\Big\}, \quad i = 1, 2, \ldots, K,$$

where $\arg\min[f_i(x, \hat{y}_i) : \hat{y}_i \in S_i(x)] = \{y_i \in S_i(x) : f_i(x, y_i) \leqslant f_i(x, \hat{y}_i), \hat{y}_i \in S_i(x)\}$. The followers observe the leader's action and simultaneously react by selecting y_i from their feasible set to minimize their objective functions, respectively.

(e) Inducible region:

$$IR = \{(x, y_1, \dots, y_K) : (x, y_1, \dots, y_K) \in S, y_i \in P_i(x), i = 1, 2, \dots, K\}.$$

To ensure that (1) has an optimal solution, we gave the following assumption.

Assumption 1

(a) S is nonempty and compact.
(b) For decisions taken by the leader, each follower has some room to respond; i.e, $P_i(x) \neq \phi$.
(c) $P_i(x)$ is a point-to-point map.

Thus in terms of the above notations, the linear BLMFP problem can be written as

$$\min\{F(x, y_1, \dots, y_K) : (x, y_1, \dots, y_K) \in IR\}. \tag{2}$$

3. Theoretical Properties and the Kth-best Approach for Linear BLMFP

THEOREM 3.1. *The inducible region can be written equivalently as a piecewise linear equality constraint comprised of supporting hyperplanes of constraint region S*

Proof. Let us begin by writing the inducible region of Definition 2(e) explicitly as follower:

$$\begin{aligned} IR = \Big\{&(x, y_1, \dots, y_K) : (x, y_1, \dots, y_K) \in S, e_i y_i = \min[e_i \tilde{y}_i : B_i \tilde{y}_i \leqslant b - Ax - \sum_{t=1, t\neq i}^{K} B_t y_t, \\ &C_i \tilde{y}_i \leqslant b_i - A_i x, C_j y_j \leqslant b_j - A_j x, j = 1, 2, \dots, K, j \neq i, \tilde{y}_i \geqslant 0], i = 1, 2, \dots, K\Big\}. \end{aligned}$$

Let us define

$$b' = (b, b_1, \dots, b_K)^T, A' = (A, A_1, \dots, A_K)^T, B_i' = (B_i, \alpha_1, \dots, \alpha_K)^T,$$

where $\alpha_j \in R^{q_j \times m_j}, i, j = 1, 2, \dots, K$ and if $j = i$, then $\alpha_j = C_i$, otherwise $\alpha_j = (0)_{q_j \times m_j}$.

Now, we have

$$IR = \{(x, y_1, \dots, y_K) : (x, y_1, \dots, y_K) \in S, e_i y_i = \min\left[e_i \tilde{y}_i : B_i' \tilde{y}_i \leqslant b_i' - A'x - \sum_{s=1, s\neq i}^{K} B_s' y_s, \tilde{y}_i \geqslant 0\right], i = 1, 2, \dots, K\}. \quad (3)$$

Let us define

$$Q_i(x, y_j, j = 1, 2, \dots K, j \neq i) = \min[e_i \tilde{y}_i : B_i' \tilde{y}_i \leqslant b_i' - A'x - \sum_{s=1, s\neq i}^{K} B_s' y_s, \tilde{y}_i \geqslant 0], \quad (4)$$

where $i = 1, 2, \dots, K$.

For each value of $x \in S(X)$, the resulting feasible region to problem (1) is nonempty and compact. Thus, for Q_i, which is a linear program parameterized in $x, y_j, j = 1, 2, \dots, K$ and $j \neq i$, always has a solution. From duality theory we get

$$\max \{u(A'x + \sum_{s=1, s\neq i}^{K} B_s' y_s - b_i') : uB_i' \geqslant -e_i, u \geqslant 0\}, \quad (5)$$

which has the same optimal value as (4) at the solution u^*. Let $u^1, \dots, u^s$ be a listing of all the vertices of the constraint region of (5) given by $U = \{u : uB_i' \geqslant -e_i, u \geqslant 0\}$. Because we know that a solution to (5) occurs at a vertex of U, we get the equivalent problem

$$\max \{u^l (A'x + \sum_{s=1, s\neq i}^{K} B_s' y_s - b_i') : u^l \in \{u^1, \dots, u^s\}\}, \quad (6)$$

which demonstrates that $Q_i(x, y_j, j = 1, 2, \dots, K, j \neq i)$, is a piecewise linear function. Rewriting IR as

$$IR = \{(x, y_1, \dots, y_k) \in S : Q_i(x, y_j, j = 1, 2, \dots, K, j \neq i) - e_i y_i = 0, i = 1, 2, \dots, K\} \quad (7)$$

yields the desired result.

COROLLARY 3.1. *The linear BLMFP problem (1) is equivalent to minimizing F over a feasible region comprised of a piecewise linear equality constraint.*

Proof. By (2) and Theorem 3.1, we have desired result. □

The each function Q_i defined by (4) is convex and continuous. In general, because we are minimizing a linear function $F = cx + \sum_{s=1}^{K} d_s y_s$ over $I\!R$, and because F is bounded below S by, say, $\min\{cx + \sum_{s=1}^{K} d_s y_s : (x, y_1, \ldots, y_K) \in S\}$, the following can be concluded.

COROLLARY 3.2. *A solution for the linear BLMFP problem occurs at a vertex of $I\!R$.*

Proof. A linear BLMFP problem can be written as in (2). Since $F = cx + \sum_{s=1}^{K} d_s y_s$ is linear, if a solution exists, one must occur at a vertex of $I\!R$. The proof is completed. □

COROLLARY 3.3. *If x is an extreme point of $I\!R$, it is an extreme point of S.*

Proof. A linear BLMFP programming can be written (2). Since $F = cx + \sum_{s=1}^{K} d_s y_s$ is linear, if a solution exists, one must occur at a vertex of $I\!R$. The proof is completed. □

THEOREM 3.2. *The solution $(x^*, y_1^*, \ldots, y_K^*)$ of the linear BLMFP problem occurs at a vertex of S*

Proof. Let $(x^1, y_1^1, \ldots, y_K^1), \ldots, (x^r, y_1^r, \ldots, y_K^r)$ be the distinct vertices of S. Since, any point in S can be written a convex combination of these vertices, let $(x^*, y_1^*, \ldots, y_K^*) = \sum_{j=1}^{r} \alpha_j (x^j, y_1^j, \ldots, y_K^j)$, where $\sum_{j=1}^{r} \alpha_j = 1, \alpha_j \geqslant 0, j = 1, 2, \ldots, \bar{r}$ and $\bar{r} \leqslant r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (1) at $(x^*, y_1^*, \ldots, y_K^*)$ in their piecewise linear form (7).

$$0 = Q_i(x, y_l^*, l = 1, 2, \ldots, K, l \neq i) - e_i y_i^*, \quad i = 1, 2, \ldots, K \tag{8}$$

Rewrite (8) as follows

$$\begin{aligned} 0 &= Q_i\Big(\sum_j \alpha_j (x^j, y_l^j, l = 1, 2, \ldots, K, l \neq i)\Big) - e_i\Big(\sum_j \alpha_j y_i^j\Big) \\ &\leqslant \sum_j \alpha_i Q_i(x^j, y_l^j, l = 1, 2, \ldots, K, l \neq i) - \sum_j \alpha_j e_i y_i^j, \end{aligned}$$

where $i = 1, 2, \ldots, K$.

By convexity of $Q_i(x, y_l, l = 1, 2, \ldots, K, l \neq i)$, we have

$$0 \leqslant \sum_j \alpha_j \big(Q_i(x^j, y_l^j, l = 1, 2, \ldots, K, l \neq i) - e_i y_i^j\big),$$

where $i = 1, 2, \ldots, K$.

But by definition,

$$Q_i(x^j, y_l^j, l=1,2,\ldots,K, l\neq i) = \min_{y_i \in S(x^j)} e_i y_i \leqslant e_i y_i^j,$$

where $i=1,2,\ldots,K$.

Therefore, $Q_i(x^j, y_l^j, l=1,2,\ldots,K, l\neq i) - e_i y_i^j \leqslant 0, j=1,2,\ldots,\bar{r}, i=1,2,\ldots,K$. Noting that $\alpha_j \geqslant 0, j=1,2,\ldots,\bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q_i(x^j, y_l^j, l=1,2,\ldots,K, l\neq i) - e_i y_i^j = 0, j=1,2,\ldots,\bar{r}, i=1,2,\ldots,K$. This implies that $(x^j, y_1^j, \ldots, y_k^j) \in I\!R$, $j=1,2,\ldots,\bar{r}$ and $(x^*, y_1^*, \ldots, y_k^*)$ can be written as a convex combination of points in $I\!R$. Because $(x^*, y_1^*, \ldots, y_k^*)$ is a vertex of $I\!R$, a contradiction results unless $\bar{r}=1$. This means that $(x^*, y_1^*, \ldots, y_k^*)$ is an extreme point of S. The proof is completed. □

Theorem 3.2 and Corollary 3.3 have provided theoretical foundation for our new algorithm. It means that by searching extreme points on the constraint region S, we can efficiently find an optimal solution for a linear BLMFP problem. The basic idea of our algorithm is that according to the objective function of the upper level, we arrange all the extreme points in S in descending order, and select the first extreme point to check if it is on the inducible region $I\!R$. If yes, the current extreme point is the optimal solution. Otherwise, the next one will be selected and checked.

More specifically, let $(x^1, y_1^1, \ldots, y_K^1), \ldots, (x^N, y_1^N, \ldots, y_K^N)$, denote the N ordered extreme points to the linear BLMFP problem

$$\min\left\{cx + \sum_{s=1}^{K} d_s y_s : (x, y_1, \ldots, y_K) \in S\right\}, \tag{9}$$

such that $cx^j + \sum_{s=1}^{K} d_s y_s^j \leqslant cx^{j+1} + \sum_{s=1}^{K} d_s y_s^{j+1}, j=1,2,\ldots,N-1$.

Let $(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_K)$ denote the optimal solution to the following problem

$$\min(f_i(x^j, y_i) : y_i \in S_i(x^j)), i=1,2,\ldots,K \tag{10}$$

We only need to find the smallest $j, j=1,2,\ldots,N$ under which $y_i^j = \tilde{y}_i$, $i=1,2,\ldots,K$.

Let us write (10) as follows

$$\begin{aligned} &\min f_i(x, y_i) \\ &\text{subject to } y_i \in S_i(x) \\ &\qquad\qquad x = x^j, \end{aligned}$$

where $i=1,2,\ldots,K$.

We only need to find the smallest j under which $y_i^j = \tilde{y}_i, i = 1, 2, \ldots, K$. From Definition 2.2(b), rewrite (10) as follows

$$\min f_i(x, y_i) = c_i x + e_i y_i \tag{11a}$$

$$\text{subject to } Ax + \sum_{t=1}^{K} B_t y_t \leqslant b \tag{11b}$$

$$A_l x + C_l y_l \leqslant b_l, \quad l = 1, 2, \ldots, K \tag{11c}$$

$$x = x^j \tag{11d}$$

$$y_1 \geqslant 0, y_2 \geqslant 0, \ldots, y_K \geqslant 0, \tag{11e}$$

where $i = 1, 2, \ldots, K$.

The solving is equivalent to select one ordered extreme point $(x^j, y_1^j, \ldots, y_K^j)$, then solve (11) to obtain the optimal solution $\tilde{y}_i$. If for all i, $y_i^j = \tilde{y}_i$, then $(x^j, y_1^j, \ldots, y_K^j)$ is the global optimum to (1). Otherwise, check the next extreme point.

It can be accomplished with the following procedure.

Step 1. Put $j \leftarrow 1$. Solve (9) with the simplex method to obtain the optimal solution $(x^1, y_1^1, \ldots, y_K^1)$. Let $W = (x^1, y_1^1, \ldots, y_K^1)$ and $T = \phi$. Go to Step 2.

Step 2. Solve (11) with the bounded simplex method. Let $\tilde{y}_i$ denote the optimal solution to (11). If $y_i^j = \tilde{y}_i$ for all $i, i = 1, \ldots, K$, stop; $(x^j, y_1^j, \ldots, y_K^j)$ is the global optimum to (1). Otherwise, go to Step 3.

Step 3. Let $W_{[j]}$ denote the set of adjacent extreme points of $(x^j, y_1^j, \ldots, y_K^j)$ such that $(x, y_1, \ldots, y_K) \in W_{[j]}$ implies $cx + \sum_{s=1}^{K} d_s y_s \leqslant cx^j + \sum_{s=1}^{K} d_s y_s^j$. Let $T = T \cup (x^j, y_1^j, \ldots, y_K^j)$ and $W = (W \cup W_{[i]}) \backslash T$. Go to Step 4.

Step 4. Set $j \leftarrow j + 1$ and choose $(x^j, y_1^j, \ldots, y_K^j)$ so that

$$cx^j + \sum_{s=1}^{K} d_s y_s^j = \min\{cx + \sum_{s=1}^{K} d_s y_s : (x, y_1 \ldots, y_K) \in W\}.$$

Go to Step 2.

4. A Numeric Example for the Kth-best Approach

Let us give the following example to show how the Kth-best approach works.

EXAMPLE 1. Consider the following linear BLMFP problem with $x \in R^1, y \in R^1, z \in R^1$ and $X = \{x \geqslant 0\}, Y = \{y \geqslant 0\}, Z = \{z \geqslant 0\}$.

$$
\begin{aligned}
&\min_{x \in X} F(x, y, z) = x - 2y - 4z \\
&\text{subject to } -x + 3y \leqslant 4 \\
&\qquad\qquad -x + z \leqslant 1 \\
&\qquad\qquad \min_{y \in Y} f_1(x, y) = x + y, \\
&\qquad\qquad \text{subject to } x - y \leqslant 0 \\
&\qquad\qquad\qquad\qquad -x - y \leqslant 0, \\
&\qquad\qquad \min_{z \in Z} f_2(x, z) = x + z \\
&\qquad\qquad \text{subject to } x + z \leqslant 4 \\
&\qquad\qquad\qquad\qquad 2x - 5z \leqslant 1 \\
&\qquad\qquad\qquad\qquad 2x + z \geqslant 1.
\end{aligned}
$$

According to the Kth-best approach, Example 1 can be rewritten as follow in the format of (9),

$$
\begin{aligned}
&\min F(x, y, z) = x - 2y - 4z \\
&\text{subject to } -x + 3y \leqslant 4 \\
&\qquad\qquad -x + z \leqslant 1 \\
&\qquad\qquad x - y \leqslant 0 \\
&\qquad\qquad -x - y \leqslant 0 \\
&\qquad\qquad x + z \leqslant 4 \\
&\qquad\qquad 2x - 5z \leqslant 1 \\
&\qquad\qquad 2x + z \geqslant 1 \\
&\qquad\qquad x \geqslant 0, y \geqslant 0, z \geqslant 0.
\end{aligned}
$$

Step 1, set $j = 1$, and solve the above problem with the simplex method to obtain the optimal solution $(x_{[1]}, y_{[1]}, z_{[1]}) = (1.5, 1.83, 2.5)$. Let $W = (1.5, 1.83, 2.5)$ and $T = \phi$. Go to Step 2.

Loop 1:

Setting $i \leftarrow 1$ and by (11), we have

$$
\begin{aligned}
&\min f_1(x, y) = x + y \\
&\text{subject to } -x + 3y \leqslant 4 \\
&\qquad\qquad -x + z \leqslant 1 \\
&\qquad\qquad x - y \leqslant 0 \\
&\qquad\qquad -x - y \leqslant 0
\end{aligned}
$$

$$\begin{aligned}
&x + z \leqslant 4 \\
&2x - 5z \leqslant 1 \\
&2x + z \geqslant 1 \\
&x = 1.5 \\
&y \geqslant 0 \\
&z \geqslant 0.
\end{aligned}$$

Using the bounded simplex method, we have $\tilde{y}_j = 1.5$. Because of $\tilde{y}_j \neq y_{[j]}$, we go to Step 3. We have
$W_{[j]} = (1.5, 1.83, 2.5), (0, 1.33, 1), (1.5, 1.5, 2.5), (0.5, 1.5, 0), (2, 2, 2)$,
$T = \{(1.5, 1.83, 2.5)\}$ and
$W = (0, 1.33, 1), (1.5, 1.5, 2.5), (0.5, 1.5, 0), (2, 2, 2)$, then go to Step 4. Update $j = 2$, and choose $(x_{[j]}, y_{[j]}, z_{[j]}) = (1.5, 1.5, 2.5)$, then go to Step 2.

Loop 2:

Setting $i \leftarrow 1$ and by (11), we have

$$\begin{aligned}
\min\ & f_1(x, y) = x + y \\
\text{subject to}\ & -x + 3y \leqslant 4 \\
& -x + z \leqslant 1 \\
& x - y \leqslant 0 \\
& -x - y \leqslant 0 \\
& x + z \leqslant 4 \\
& 2x - 5z \leqslant 1 \\
& 2x + z \geqslant 1 \\
& x = 1.5 \\
& y \geqslant 0 \\
& z \geqslant 0.
\end{aligned}$$

Using the bounded simplex method, we have $\tilde{y}_j = 1.5$ and $\tilde{y}_j = y_{[j]}$. Setting $i \leftarrow i + 1$ and by (11), we have

$$\begin{aligned}
\min\ & f_2(x, z) = x + z \\
\text{subject to}\ & -x + 3y \leqslant 4 \\
& -x + z \leqslant 1 \\
& x - y \leqslant 0 \\
& -x - y \leqslant 0 \\
& x + z \leqslant 4
\end{aligned}$$

$$\begin{aligned}&2x-5z\leqslant 1\\&2x+z\geqslant 1\\&x=1.5\\&y\geqslant 0\\&z\geqslant 0.\end{aligned}$$

Using the bounded simplex method, we have $\tilde{z}_j=0.4$. Because of $\tilde{z}_j\neq z_{[j]}$, we go to Step 3. We have
$W_{[j]}=\{(1.5, 1.83, 2.5), (0, 0, 1), (1.5, 1.5, 2.5), (0.5, 0.5, 0), (2, 2, 2)\}$,
$T=\{(1.5, 1.83, 2.5), (1.5, 1.5, 2.5)\}$ and
$W=\{(0, 1.33, 1), (0.5, 1.5, 0), (2, 2, 2), (0, 0, 1), (0.5, 0.5, 0)\}$, then go to Step 4. Update $j=3$, and choose $(x_{[j]}, y_{[j]}, z_{[j]})=(2, 2, 2)$, then go to Step 2.

Loop 3:
Setting $i\leftarrow 1$ and by (11), we have

$$\begin{aligned}\min\ & f_1(x,y)=x+y\\ \text{subject to}\ & -x+3y\leqslant 4\\&-x+z\leqslant 1\\&x-y\leqslant 0\\&-x-y\leqslant 0\\&x+z\leqslant 4\\&2x-5z\leqslant 1\\&2x+z\geqslant 1\\&x=2\\&y\geqslant 0\\&z\geqslant 0.\end{aligned}$$

Using the bounded simplex method, we have $\tilde{y}_j=2$ and $\tilde{y}_j=y_{[j]}$. Setting $i\leftarrow i+1$ and by (11), we have

$$\begin{aligned}\min\ & f_2(x,z)=x+z\\ \text{subject to}\ & -x+3y\leqslant 4\\&-x+z\leqslant 1\\&x-y\leqslant 0\\&-x-y\leqslant 0\\&x+z\leqslant 4\end{aligned}$$

$$\begin{aligned}
&2x - 5z \leqslant 1\\
&2x + z \geqslant 1\\
&x = 2\\
&y \geqslant 0\\
&z \geqslant 0.
\end{aligned}$$

Using the bounded simplex method, we have $\tilde{z}_j = 0.6$. Because of $\tilde{z}_j \neq z_{[j]}$, we go to Step 3. We have
$W_{[j]} = \{(1.5, 1.83, 2.5), (1.5, 1.5, 2.5), (2, 2, 0.6), (2, 2, 2)\}$,
$T = \{(1.5, 1.83, 2.5), (1.5, 1.5, 2.5), (2, 2, 2)\}$ and
$W = \{(0, 1.33, 1), (0.5, 1.5, 0), (0, 0, 1), (0.5, 0.5, 0), (2, 2, 0.6)\}$, then go to Step 4.
Update $j = 3$, and choose $(x_{[j]}, y_{[j]}, z_{[j]}) = (2, 2, 0.6)$, then go to Step 2.

Loop 4:

Setting $i \leftarrow 1$ and by (11), we have

$$\begin{aligned}
\min f_1(x, y) = x + y&\\
\text{subject to } &-x + 3y \leqslant 4\\
&-x + z \leqslant 1\\
&x - y \leqslant 0\\
&-x - y \leqslant 0\\
&x + z \leqslant 4\\
&2x - 5z \leqslant 1\\
&2x + z \geqslant 1\\
&x = 2\\
&y \geqslant 0\\
&z \geqslant 0
\end{aligned}$$

Using the bounded simplex method, we have $\tilde{y}_j = 2$ and $\tilde{y}_j = y_{[j]}$. Setting $i \leftarrow i + 1$ and by (11), we have

$$\begin{aligned}
\min f_2(x, z) = x + z&\\
\text{subject to } &-x + 3y \leqslant 4\\
&-x + z \leqslant 1\\
&x - y \leqslant 0\\
&-x - y \leqslant 0\\
&x + z \leqslant 4
\end{aligned}$$

$$
\begin{aligned}
&2x - 5z \leqslant 1\\
&2x + z \geqslant 1\\
&x = 2\\
&y \geqslant 0\\
&z \geqslant 0.
\end{aligned}
$$

Using the bounded simplex method, we have $\tilde{z}_j = 0.6$ and $\tilde{z}_j = z_{[j]}$. The solution $(x_{[j]}, y_{[j]}, z_{[j]}) = (2, 2, 0.6)$ is the global solution to the example. Therefore, the optimal solution of the bilevel multi-follower problem occurs at the point $(x^*, y^*, z^*) = (2, 2, 0.6)$ with the leader's objective value $F^* = -4.4$, and two followers' objective values $f_1^* = 4$ and $f_2^* = 2.6$, respectively.

5. Computational Test for the *K*th-best Approach

No computational experience was reported for the Kth-best algorithm for one leader and one follower linear bilevel problems. We used following two BLMFP problems to test our Kth-best algorithm.

EXAMPLE 1.

$$
\begin{aligned}
&\min_{x_1, x_2 \in X} F(x_1, x_2, y, z) = 3x_1 + 8x_2 + 7y + 11z\\
&\text{subject to } 5x_1 + 2x_2 - y + 6z \leqslant 40\\
&\qquad 6x_1 - x_2 + 13y \leqslant 15\\
&\qquad x_1 + x_2 - 7z \leqslant 10\\
&\qquad 7y + 4z \leqslant 20\\
&\qquad \min_{y \in Y} f_1(x_1, x_2, y) = 2x_1 + x_2 - y\\
&\qquad \text{subject to } 5x_1 + 7y \leqslant 15\\
&\qquad\qquad -4x_2 + 25y \leqslant 3\\
&\qquad \min_{z \in Z} f_2(x_1, x_2, z) = 15x_1 - x_2 + 80z\\
&\qquad \text{subject to } 40x_1 + z \leqslant 5,
\end{aligned}
$$

where $x_1, x_2 \in R^1$, $y \in R^1$, $z \in R^1$ and $X = \{x_1 > 0, x_2 > 0\}$, $Y = \{y > 0\}$, $Z = z > 0$.

The CPU run time is 0.08799 sec, loop times are 7 and an optimal solution occurs at the point $(x_1^*, x_2^*, y^*, z^*) = (0.12, 11.79, 2.01, 0.27)$ with $F^* = 111.69$, $f_1^* = 10.02$ and $f_2^* = 11.61$.

EXAMPLE 2.

$$
\begin{aligned}
\min_{x_1 \in X} F(x_1, x_2, \ldots, x_{11}) = {} & 80x_1 - 0.5x_2 + 3x_3 + 0.7x_4 - 10x_5 - 0.99x_6 \\
& + 13x_7 + 31x_8 - 0.3x_9 + 0.4x_{10} - 0.23x_{11}
\end{aligned}
$$

$$
\begin{aligned}
\text{subject to } & x_1 + x_2 \leqslant 18.5 \\
& -0.31x_1 + x_3 \geqslant 0 \\
& 3.2x_1 - x_4 \geqslant 0 \\
& x_1 + x_5 \geqslant 5.5 \\
& -0.25x_1 - x_6 \leqslant 0 \\
& 4.6x_1 - x_7 \geqslant 0 \\
& 0.5x_1 + 0.48x_8 \leqslant 11 \\
& 4.34x_1 - x_9 \geqslant 0 \\
& 0.32x_1 - x_{10} \leqslant 0 \\
& x_1 + x_{11} \geqslant 7.9 \\
& \min_{x_2 \in X} f_1(x_1, x_2) = 0.9x_1 + 3x_2 \\
& \text{subject to } 2.1x_1 - x_2 \geqslant 0 \\
& \qquad\qquad\quad x_1 + x_2 \geqslant 2.7 \\
& \min_{x_3 \in X} f_2(x_1, x_3) = 3x_1 - 2.2x_3 \\
& \text{subject to } x_1 + x_3 \leqslant 20 \\
& \qquad\qquad\quad 0.8x_1 + 0.91x_3 \geqslant 1.8 \\
& \min_{x_4 \in X} f_3(x_1, x_4) = 0.4x_1 - x_4 \\
& \text{subject to } x_1 + x_4 \leqslant 31.5 \\
& \qquad\qquad\quad -0.4x_1 + x_4 \geqslant 0 \\
& \min_{x_5 \in X} f_4(x_1, x_5) = 0.7x_1 + 21x_5 \\
& \text{subject to } 0.3x_1 + 0.4x_5 \leqslant 8.5 \\
& \qquad\qquad\quad -2.8x_1 + x_5 \leqslant 0 \\
& \min_{x_6 \in X} f_5(x_1, x_6) = 10x_1 + 0.67x_6 \\
& \text{subject to } x_1 + x_6 \leqslant 31.4 \\
& \qquad\qquad\quad x_1 + x_6 \geqslant 6.3 \\
& \min_{x_7 \in X} f_6(x_1, x_7) = 2x_1 - 3x_7 \\
& \text{subject to } 2x_1 + x_7 \leqslant 17.5 \\
& \qquad\qquad\quad -0.5x_1 + x_7 \geqslant 0
\end{aligned}
$$

$$\begin{aligned}
&\min_{x_8 \in X} f_7(x_1, x_8) = 0.75x_1 - 20.5x_8 \\
&\text{subject to } -8.6x_1 + x_8 \leqslant 0 \\
&\qquad\qquad x_1 + x_8 \geqslant 7.6 \\
&\min_{x_9 \in X} f_8(x_1, x_9) = 0.3x_1 + 6.7x_9 \\
&\text{subject to } x_1 + x_9 \leqslant 18.6 \\
&\qquad\qquad 0.26x_1 - x_9 \leqslant 0 \\
&\min_{x_{10} \in X} f_9(x_1, x_{10}) = 0.65x_1 - 3.2x_{10} \\
&\text{subject to } x_1 + x_{10} \leqslant 16.7 \\
&\qquad\qquad x_1 + x_{10} \geqslant 5.85 \\
&\min_{x_{12} \in X} f_{10}(x_1, x_{11}) = x_1 + 0.56x_{11} \\
&\text{subject to } x_1 + x_{11} \leqslant 70.5 \\
&\qquad\qquad -0.15x_1 + x_{11} \geqslant 0,
\end{aligned}$$

where $x_1, x_2, \ldots, x_{11} \in R^1$, and $X = \{x_1 > 0, x_2 > 0, \ldots, x_{11}\}$.

The CPU run time is 1.12914 sec, loop times are 34 and an optimal solution occurs at the point $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*, x_9^*, x_{10}^*, x_{11}^*) =$ (1.45, 1, 25, 18, 55, 4.63, 4.05, 4.85, 6.66, 12.45, 0.38, 15.25, 0.22) with $F^* = 607.44$, $f_1^* = 5.06$, $f_2^* = -36.46$, $f_3^* = -4.05$, $f_4^* = 86.06$, $f_5^* = 17.75$, $f_6^* = -17.08$, $f_7^* = -254.14$, $f_8^* = 2.98$, $f_9^* = -47.86$ and $f_{10}^* = 1.57$.

All computations were performed on an Intel® Pentium® 4, CPU 2.8 GHz and 512 MB RAM. Web server is ISS 5. Database Server is Microsoft SQL Server 2000. The application is developed using. ASP. Table 1 shows the computational results for the above examples.

As expected, the CPU time grew exponentially with the size of the problem, more importantly, the number loops depended on the size of the problem too.

Table 1. Computational results

	Number followers	Number constraints	CPU time (sec.)	Number loops	CPU time/Loop (sec/loop)
Example 1	2	11	0.08799	7	0.01257
Example 2	10	41	1.12914	34	0.03321

6. Conclusion and Further Study

This paper addresses the theoretical properties of linear bilevel multi-follower programming problems in which there are no sharing variables except the leader's. This paper also presents the Kth-best approach for linear bilevel multi-follower programming and gives a computational test for this approach. The further study of the research is to explore theoretical properties of linear bilevel multi-follower programming problems in which there are sharing variables among followers.

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